ON A BOUNDARY VALUE PROBLEM IN SUBSONIC AEROELASTICITY AND THE COFINITE HILBERT TRANSFORM.

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Abstract. We study a boundary value problem in subsonic aeroelasticity and introduce the *cofinite Hilbert transform* as a tool in solving an auxiliary linear integral equation on the complement of a finite interval on the real line $\mathbb{R}$.

1. Introduction.

We consider the linearized subsonic inviscid compressible flow equation in 2D ([BAH], [Ba2])

$$a_\infty^2\left(1-M^2\right)\frac{\partial^2\phi}{\partial x^2}+a_\infty^2\frac{\partial^2\phi}{\partial z^2}=\frac{\partial^2\phi}{\partial t^2}+2Ma_\infty\frac{\partial^2\phi}{\partial t\partial x}, \tag{1}$$

where a_∞ is the speed of sound, $M=\dfrac{U}{a_\infty}<1$ - the Mach number, U - free stream velocity, $\phi(x,z,t)$ - small disturbance velocity potential, considered on

$$\mathbb{R}_+^2\times\overline{\mathbb{R}_+}=\left\{(x,z,t): -\infty<x<\infty,\ 0<z<\infty,\ 0\le t<\infty\right\},$$

with boundary conditions:

- flow tangency condition

$$\frac{\partial\phi}{\partial z}(x,0,t)=w_a(x,t),\ |x|<b, \tag{2}$$

  where b is the "half-chord", and w_a is the given normal velocity of the wing, without loss of generality we will assume in what follows that $b=1$,
- *"strong Kutta-Joukowski condition"* for the acceleration potential

$$\psi(x,z,t):=\frac{\partial\phi}{\partial t}+U\frac{\partial\phi}{\partial x},$$

$$\psi(x,0,t)=0 \text{ for } 1<|x|<A \text{ for some } A>1, \tag{3}$$

- far field condition

$$\phi(x,z,t)\to 0,\ \text{as } |x|\to\infty,\ \text{or } z\to\infty.$$

Boundary condition (3), though being motivated by one of the "auxiliary boundary conditions" from ([BAH], p. 319), is weaker, because it requires that $\psi(x,0,t)=0$ not on the whole $\mathbb{R}\setminus[-1,1]$, but only on finite intervals adjacent to the interval $[-1,1]$. On the other hand this change in the boundary condition allows application of some new mathematical tools different from tools in [BAH] and [Ba2].

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In order to formulate our main result we introduce the following notations. We denote by $\widehat{w_a}$ the Laplace transform of the function w_a with respect to time variable

$$\widehat{w_a}(x,z,\lambda) = \int_0^\infty e^{-\lambda t} w_a(x,z,t)dt$$

for $\mathrm{Re}\lambda > \sigma_a > 0$. We also denote

$$r(\lambda) = \frac{\lambda M}{U\sqrt{1-M^2}},$$

$$d(\lambda) = \frac{\lambda M^2}{U(1-M^2)}.$$

In sections 5 and 6 we construct a function $\mathcal{D}_N(\lambda)$ (equation (41)), analytic in the half-plane $\mathrm{Re}\lambda > \sigma_a > 0$, and depending only on the function K_0 - the modified Bessel function of the third kind.

The following theorem represents the main result of the paper.

Theorem 1. *Let function $\mathcal{D}_N(\lambda)$ from equation (41), mentioned above, have no zeros in the strip $\{Re\lambda \in [\sigma_1, \sigma_2]\}$, where $\sigma_1 > \sigma_a$. Let $I(1) = [-1, 1]$, and let $w_a(\cdot, t) \in L^2(I(1))$ be such that for some $\epsilon > 0$*

$$\|\widehat{w}_a(\cdot, \sigma + i\eta)\|_{L^2(I(1))} < \exp\left\{-e^{|\eta|}\cdot(1+|\eta|)^{2+\epsilon}\right\} \text{ for } \sigma \in [\sigma_1, \sigma_2] \tag{4}$$

Then equation (1) has a solution of the form

$$\begin{aligned}\phi(x,z,t) = &-\frac{1}{2\pi\sqrt{1-M^2}}\int_{\sigma'-i\infty}^{\sigma'+i\infty} e^{d(\sigma'+i\eta)x} \\ &\times\left[\int_{-\infty}^{\infty} K_0\left(r(\sigma'+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right)h_a(y,\sigma'+i\eta)dy\right]e^{(\sigma'+i\eta)t}d\eta.\end{aligned} \tag{5}$$

This solution is independent of $\sigma' \in [\sigma_1, \sigma_2]$, satisfies boundary conditions above, and function h_a satisfies the estimate

$$\int_{-\infty}^{\infty}(1+|x|)^{p-2}\left|h_a(x,\sigma'+i\eta)\right|^p dx < \frac{C}{(1+|\eta|)^m}$$

for arbitrary $m > 0$, $p < \dfrac{4}{3}$, and $C > 0$ independent of λ.

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2. "General" solution.

We are seeking a solution of the equation (1) of the form

$$\phi(x,z,t) = \int_{\sigma-i\infty}^{\sigma+i\infty}\xi(x,z,\lambda)e^{(\sigma+i\eta)t}d\eta, \tag{6}$$

where $\lambda = \sigma + i\eta$, $\sigma > \sigma_a$ and $\xi(x,z,\lambda) \in L^1\eta(\mathbb{R})$. Then, substituting the expression above into equation (1), we obtain the following auxiliary equation for ξ

$$a_\infty^2\left(1-M^2\right)\frac{\partial^2\xi}{\partial x^2} + a_\infty^2\frac{\partial^2\xi}{\partial z^2} - \lambda^2\xi - 2M\lambda a_\infty\frac{\partial\xi}{\partial x} = 0. \tag{7}$$

To describe the general solution of equation (7) satisfying the far field condition we consider, following [Ba2]

$$D(\omega,\lambda) = M^2\left(\frac{\lambda}{U}\right)^2 + 2i\frac{\lambda}{U}M^2\omega + \left(1-M^2\right)\omega^2,$$

and prove two lemmas below.

Lemma 2.1. *There exists a function $\sqrt{D(\omega,\lambda)}$, analytic with respect to complex variable $\frac{\lambda}{U}+i\omega$ ($\frac{\lambda}{U}\in\mathbb{C},\ \omega\in\mathbb{R}$) in the half-plane Re$\lambda>\sigma_a$, and such that Re$\sqrt{D(\omega,\lambda)}>0$.*

Proof. Representing $D(\omega,\lambda)$ as

$$D(\omega,\lambda)=M^2\left(\frac{\lambda}{U}\right)^2+2i\frac{\lambda}{U}M^2\omega+\left(1-M^2\right)\omega^2=M^2\left(\frac{\lambda}{U}+i\omega\right)^2+\omega^2,$$

we obtain that the image of the half-plane Re$\lambda>\sigma_a$ under the map $D(\omega,\lambda)$ is contained in the domain $\mathbb{C}\setminus\mathbb{R}^-$. Then the branch of the function $\sqrt{}$ considered on the complex plane with the cut along the negative part of the real axis is well defined and analytic on the image of D, and its real part satisfies condition of the Lemma. Therefore, the composition $\sqrt{D}$ is also analytic and satisfies the same condition. $\square$

Lemma 2.2. *The following equality holds*

$$\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}K_0\left(r(\lambda)\left(\frac{x^2}{1-M^2}+z^2\right)^{\frac12}\right)=\mathcal{F}\left[\frac{e^{-z\sqrt{D(\omega,\lambda)}}}{2\sqrt{D(\omega,\lambda)}}\right],$$

where $\mathcal{F}$ denotes the Fourier transform, or

$$\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}K_0\left(r(\lambda)\left(\frac{x^2}{1-M^2}+z^2\right)^{\frac12}\right)$$
$$=\int_{-\infty}^{\infty}e^{ix\omega}\frac{e^{-z\left((1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)\right)^{\frac12}}}{2\sqrt{(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)}}d\omega.$$

Proof. First, we represent $D\left(\omega,\lambda\right)$ as

$$D\left(\omega,\lambda\right)=\left(1-M^2\right)\omega^2+2i\frac{\lambda}{U}M^2\omega+M^2\left(\frac{\lambda}{U}\right)^2$$
$$=\left(\omega\sqrt{1-M^2}+i\frac{\lambda M^2}{U\sqrt{1-M^2}}\right)^2+\left(\frac{\lambda M^2}{U\sqrt{1-M^2}}\right)^2+M^2\left(\frac{\lambda}{U}\right)^2$$
$$=(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda).$$

Changing variables in equality ([EMOT])

$$K_0\left(r(x^2+z^2)^{\frac12}\right)=\int_{-\infty}^{\infty}e^{ixu}\frac{e^{-z\left(u^2+r^2\right)^{\frac12}}}{2\sqrt{u^2+r^2}}du,$$

we obtain

$$K_0\left(r(x^2+z^2)^{\frac12}\right)=\int_{-\infty}^{\infty}e^{ix\sqrt{1-M^2}\omega}\frac{e^{-z\left((1-M^2)\omega^2+r^2\right)^{\frac12}}}{2\sqrt{(1-M^2)\omega^2+r^2}}d\left(\sqrt{1-M^2}\omega\right),$$

and then

$$\frac{1}{\sqrt{1-M^2}}K_0\left(r\left(\frac{x^2}{1-M^2}+z^2\right)^{\frac12}\right)=\int_{-\infty}^{\infty}e^{ix\omega}\frac{e^{-z\left((1-M^2)\omega^2+r^2\right)^{\frac12}}}{2\sqrt{(1-M^2)\omega^2+r^2}}d\omega.$$

We transform the equality above by integrating function

$$g(x,w) = e^{ixw}\frac{e^{-z\left((1-M^2)w^2+r^2\right)^{\frac{1}{2}}}}{\sqrt{(1-M^2)w^2+r^2}}, \quad w \in \mathbb{C},$$

analytic with respect to w, over the piecewise linear contour

$$[-C, C, C+id, -C+id] \in \mathbb{C}, \quad \text{with } C \in \mathbb{R}, C > 0,\ d \in \mathbb{C},\ \mathrm{Re}d > 0.$$

Then we obtain

$$\int_{-C}^{C} g(x,w)dw + \int_{C}^{C+id} g(x,w)dw + \int_{C+id}^{-C+id} g(x,w)dw + \int_{-C+id}^{-C} g(x,w)dw = 0. \tag{8}$$

For C large enough we have the following estimates for $w = u + iv \in [-C, -C+id]$, and $w \in [C, C+id]$

$$\left|e^{ix(u+iv)}\right| < e^{|x|\cdot\mathrm{Re}d}, \quad \left|\sqrt{(1-M^2)w^2+r^2}\right| > \sqrt{1-M^2}\frac{C}{2},$$

$$\left|e^{-z\left((1-M^2)w^2+r^2\right)^{\frac{1}{2}}}\right| < e^{-z\sqrt{1-M^2}\frac{C}{2}},$$

and therefore for $z > 0$

$$\left|\int_{C}^{C+id} g(x,w)dw\right|, \left|\int_{-C}^{-C+id} g(x,w)dw\right| \to 0 \ \text{ as } C \to \infty.$$

Using the last estimate in (8) we obtain equality

$$\int_{-\infty}^{\infty} e^{ix\omega}\frac{e^{-z\left((1-M^2)\omega^2+r^2(\lambda)\right)^{\frac{1}{2}}}}{\sqrt{(1-M^2)\omega^2+r^2(\lambda)}}d\omega$$

$$= \int_{-\infty}^{\infty} e^{ix(\omega+id(\lambda))}\frac{e^{-z\left((1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)\right)^{\frac{1}{2}}}}{\sqrt{(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)}}d(\omega+id(\lambda)),$$

and finally

$$\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}K_0\left(r(\lambda)\left(\frac{x^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right) \tag{9}$$

$$= \int_{-\infty}^{\infty} e^{ix\omega}\frac{e^{-z\left((1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)\right)^{\frac{1}{2}}}}{2\sqrt{(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)}}d\omega.$$

$\square$

Using now Lemmas 2.1 and 2.2 we consider a special representation of the general solution of (7). Namely, using notations of Lemma 2.2, and denoting

$$S(x,z,\lambda) = -\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}K_0\left(r(\lambda)\left(\frac{x^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right),$$

we consider

$$\begin{aligned}\xi(x,z,\lambda) &= \int_{-\infty}^{\infty} S(x-y,z,\lambda)v_a(y,\lambda)dy \\ &= -\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}\int_{-\infty}^{\infty} e^{-d(\lambda)y}K_0\left(r(\lambda)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right)v_a(y,\lambda)dy.\end{aligned} \tag{10}$$

Proposition 2.3. *Function ξ defined by formula (10) satisfies equation (7). If*

$$\xi(x,z,\lambda),\ \frac{\partial^2\xi}{\partial x^2}(x,z,\lambda),\ \frac{\partial^2\xi}{\partial z^2}(x,z,\lambda),\ |\eta|^2\xi(x,z,\lambda),\ |\eta|\frac{\partial\xi(x,z,\lambda)}{\partial x} \in L^1(\mathbb{R}_\eta),$$

where $\lambda=\sigma+i\eta$, then the inverse Laplace transform of ξ, defined by the formula ([Boc]*)*

$$\phi(x,z,t) = \frac{1}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{(\sigma+i\eta)t}\xi(x,z,\sigma+i\eta)d\eta \tag{11}$$

satisfies equation (1).

Proof. To prove that ξ defined above satisfies equation (7) it suffices to prove that function S satisfies the same equation. For S we have

$$\begin{aligned}&a_\infty^2\left(1-M^2\right)\frac{\partial^2 S}{\partial x^2}+a_\infty^2\frac{\partial^2 S}{\partial z^2}-\lambda^2 S-2M\lambda a_\infty\frac{\partial S}{\partial x} \\ &= a_\infty^2\left[\frac{\partial^2 S}{\partial z^2}+\left(1-M^2\right)\frac{\partial^2 S}{\partial x^2}-M^2\left(\frac{\lambda}{U}\right)^2 S-2M^2\frac{\lambda}{U}\frac{\partial S}{\partial x}\right].\end{aligned}$$

Using then formula (9), we obtain

$$\begin{aligned}&\frac{\partial^2 S}{\partial z^2}+\left(1-M^2\right)\frac{\partial^2 S}{\partial x^2}-M^2\left(\frac{\lambda}{U}\right)^2 S-2M^2\frac{\lambda}{U}\frac{\partial S}{\partial x} \\ &= -\int_{-\infty}^{\infty} e^{ix\omega}\left((1-M^2)(\omega+id)^2+r^2\right)\frac{e^{-z\left((1-M^2)(\omega+id)^2+r^2\right)^{\frac{1}{2}}}}{2\sqrt{(1-M^2)(\omega+id)^2+r^2}}d\omega \\ &\quad+\int_{-\infty}^{\infty} e^{ix\omega}\left(1-M^2\right)\omega^2\frac{e^{-z\left((1-M^2)(\omega+id)^2+r^2\right)^{\frac{1}{2}}}}{2\sqrt{(1-M^2)(\omega+id)^2+r^2}}d\omega \\ &\quad+\int_{-\infty}^{\infty} e^{ix\omega}M^2\left(\frac{\lambda}{U}\right)^2\frac{e^{-z\left((1-M^2)(\omega+id)^2+r^2\right)^{\frac{1}{2}}}}{2\sqrt{(1-M^2)(\omega+id)^2+r^2}}d\omega \\ &\quad+\int_{-\infty}^{\infty} e^{ix\omega}2M^2\frac{\lambda}{U}i\omega\frac{e^{-z\left((1-M^2)(\omega+id)^2+r^2\right)^{\frac{1}{2}}}}{2\sqrt{(1-M^2)(\omega+id)^2+r^2}}d\omega = 0.\end{aligned}$$

To prove that function ϕ defined by formula (11) satisfies equation (1) we apply the inverse Laplace transform to equality

$$a_\infty^2\left(1-M^2\right)\frac{\partial^2\xi}{\partial x^2}+a_\infty^2\frac{\partial^2\xi}{\partial z^2}-\lambda^2\xi-2M\lambda a_\infty\frac{\partial\xi}{\partial x}=0$$

and obtain equation (1) for ϕ. ∎

3. Boundary Conditions.

In this section we reformulate the boundary conditions of section 1 in terms of function $v_a(y,\lambda)$ from formula (10).

To check the flow tangency condition (2) we use formulas (10) and (9), and obtain

$$\frac{\partial}{\partial z}\xi(x,z,\lambda)\Big|_{z=0} = \frac{\partial}{\partial z}\int_{-\infty}^{\infty} S(x-y,z,\lambda)v_a(y,\lambda)dy\Big|_{z=0} \tag{12}$$

$$= -\frac{\partial}{\partial z}\int_{-\infty}^{\infty} v_a(y,\lambda)dy\int_{-\infty}^{\infty} e^{i(x-y)\omega}\frac{e^{-z\left((1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)\right)^{\frac{1}{2}}}}{2\sqrt{(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)}}d\omega\Big|_{z=0}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{ix\omega}d\omega\int_{-\infty}^{\infty} e^{-iy\omega}v_a(y,\lambda)dy = \pi\cdot v_a(x,\lambda),$$

which, after comparison with equality (2) leads to a unique choice

$$v_a(x,\lambda) = \frac{1}{\pi}\widehat{w_a}(x,\lambda) \text{ for } |x|<1. \tag{13}$$

To satisfy the Kutta-Joukowski boundary condition (3) we should have

$$\left(\frac{\partial\phi}{\partial t}+U\frac{\partial\phi}{\partial x}\right)\Big|_{z=0} = 0 \text{ for } 1<|x|<A,$$

or equality

$$\lambda\xi(x,0,\lambda)+U\frac{\partial\xi}{\partial x}(x,0,\lambda) = 0 \text{ for } 1<|x|<A$$

for function ξ.

Substituting ξ from formula (10) into equality above we obtain the following condition for $1<|x|<A$:

$$0 = \left(\lambda+U\frac{\partial}{\partial x}\right)\xi(x,0,\lambda)$$

$$= -\left(\lambda+U\frac{\partial}{\partial x}\right)\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}\int_{-\infty}^{\infty} e^{-d(\lambda)y}K_0\left(r(\lambda)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right)v_a(y,\lambda)dy\Big|_{z=0}. \tag{14}$$

To reformulate the last condition as an integral equation we use condition (13), and define

$$g_a(x,\lambda) = \frac{e^{d(\lambda)x}}{\pi}\int_{-1}^{1} e^{-d(\lambda)y}R(x-y,\lambda)\widehat{w_a}(y,\lambda)dy \text{ for } 1<|x|<A,$$

with kernel $R(x,\lambda)$ defined by the formula

$$R(x,\lambda) = \left[(\lambda+Ud(\lambda))\,K_0\left(\frac{r(\lambda)|x|}{\sqrt{1-M^2}}\right)+U\frac{\partial}{\partial x}K_0\left(\frac{r(\lambda)|x|}{\sqrt{1-M^2}}\right)\right]. \tag{15}$$

Then condition (14) will be satisfied if v_a will satisfy the following integral equation

$$e^{d(\lambda)x}\int_{|y|>1} e^{-d(\lambda)y}R(x-y,\lambda)v_a(y,\lambda)dy = -g_a(x,\lambda) \text{ for } 1<|x|<A.$$

Further simplifying the equation above we consider $h_a(y,\lambda) := e^{-d(\lambda)y}\cdot v_a(y,\lambda)$ as an unknown function, and rewrite it as

$$\int_{|y|>1} R(x-y,\lambda)h_a(y,\lambda)dy = f_a(x,\lambda) \text{ for } 1<|x|<A, \tag{16}$$

where $f_a(x,\lambda) = -e^{-d(\lambda)x}\cdot\chi_A(x)g_a(x,\lambda)$ is defined for

$$\{(x,\lambda)\in\mathbb{R}\times\mathbb{C} : |x|>1, \mathrm{Re}\lambda\in[\sigma_1,\sigma_2]\}$$

by the formula

$$f_a(x,\lambda) = -\frac{\chi_A(x)}{\pi}\int_{-1}^{1} e^{-d(\lambda)y} R(x-y,\lambda)\widehat{w_a}(y,\lambda)dy \tag{17}$$

with

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in [-A,A]\setminus[-1,1], \\ 0 & \text{otherwise.} \end{cases}$$

4. Cofinite Hilbert transform.

As a first step in the analysis of solvability of (16) we prove solvability for the operator, closely related to operator $\mathcal{R}_\lambda$ from (16), and which in analogy with the Tricomi's definition of the finite Hilbert transform [Tr] we call the *cofinite Hilbert transform*.

We define the cofinite Hilbert transform on the set of functions on

$$I^c(1) = \mathbb{R}\setminus[-1,1]$$

by the formula

$$\mathcal{P}[h](x) = \frac{1}{\pi}\int_{|y|>1}\frac{h(y)}{y-x}dy \text{ for } |x|>1, \tag{18}$$

where the integral

$$\int_{|y|>1} = \int_{-\infty}^{-1} + \int_{1}^{\infty}$$

is understood in the sense of Cauchy's principal value.

In the proposition below we prove solvability for the nonhomogeneous integral equation with operator $\mathcal{P}$ in weighted spaces

$$\mathcal{L}^p\left(I^c(1)\right) = \left\{ f : \int_{|x|>1} |x|^{p-2}\left|f(x)\right|^p dx < \infty \right\}$$

with

$$\|f\|_{\mathcal{L}^p(I^c(1))} = \left(\int_{|x|>1} |x|^{p-2}\left|f(x)\right|^p dx\right)^{1/p}.$$

Proposition 4.1. *For any function $f \in \mathcal{L}^q\left(I^c(1)\right)$ with $q > \dfrac{4}{3}$ there exists a solution h of equation*

$$\mathcal{P}[h] = f, \tag{19}$$

such that $h \in \mathcal{L}^p\left(I^c(1)\right)$ for any $p < \dfrac{4}{3}$.

Proof. We consider the following diagram of transformations

$$\begin{array}{ccc} L^p\left(I(1)\right) & \overset{-\mathcal{T}}{\rightarrow} & L^p\left(I(1)\right) \\ \downarrow \Theta & & \downarrow \Theta \\ \mathcal{L}^p\left(I^c(1)\right) & \overset{\mathcal{P}}{\rightarrow} & \mathcal{L}^p\left(I^c(1)\right), \end{array} \tag{20}$$

where $\mathcal{T}$ is the finite Hilbert transform, $\mathcal{P}$ is the cofinite Hilbert transform, and

$$\Theta : L^p\left(I(1)\right) \to \mathcal{L}^p\left(I^c(1)\right)$$

is defined by the formula

$$\Theta[f](x) = \frac{1}{x} f\left(\frac{1}{x}\right). \tag{21}$$

To prove that the maps in diagram (20) are well defined we use equality

$$\|\Theta[f]\|^p_{\mathcal{L}^p} = \int_{|x|>1} |x|^{p-2}\, |\Theta[f](x)|^p\, dx = \int_{|x|>1} |x|^{p-2} \frac{\left|f\left(\frac{1}{x}\right)\right|^p}{|x|^p} dx$$
$$= -\int_1^{-1} |f\,(t)|^p\, dt = \|f\|^p_p,$$

and notice that for

$$\Theta^* : \mathcal{L}^p\left(I^c(1)\right) \to L^p\left(I(1)\right)$$

defined by the same formula

$$\Theta^*[f](x) = \frac{1}{x} f\left(\frac{1}{x}\right),$$

we have

$$\Theta \circ \Theta^*\,[f]\,(x) = \Theta\left[\frac{1}{y} f\left(\frac{1}{y}\right)\right](x) = \frac{1}{x} \cdot x f(x) = f(x). \tag{22}$$

Diagram (20) is commutative, as can be seen from equality

$$\mathcal{P}\left[\Theta[f]\right](x) = \frac{1}{\pi}\int_{|y|>1} \frac{f\left(\frac{1}{y}\right)}{y(y-x)} dy = \frac{1}{\pi}\int_{-1}^{1} t \frac{f\,(t)}{(\frac{1}{t}-x)t^2} dt$$
$$= \frac{1}{\pi x}\int_{-1}^{1} \frac{f\,(t)}{\frac{1}{x}-t} dt = \Theta\left[-\mathcal{T}[f]\right].$$

To "invert" operator $\mathcal{P}$ we use commutativity of diagram (20), relation (22), and operator ([So],[Tr])

$$\mathcal{T}^{-1} : L^{\frac{4}{3}+}\left(I(1)\right) \to L^{\frac{4}{3}-}\left(I(1)\right),$$

defined by the formula

$$\mathcal{T}^{-1}[g](x) = -\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1-y^2}{1-x^2}} \frac{g(y)}{y-x} dy,$$

and satisfying

$$\mathcal{T} \circ \mathcal{T}^{-1}[f] = f.$$

Namely, we define operator

$$\mathcal{P}^{-1} : \mathcal{L}^{\frac{4}{3}+}\left(I^c(1)\right) \to \mathcal{L}^{\frac{4}{3}-}\left(I^c(1)\right)$$

by the formula

$$\mathcal{P}^{-1}[f] = -\Theta \circ \mathcal{T}^{-1} \circ \Theta^*[f].$$

Then

$$\mathcal{P} \circ \mathcal{P}^{-1}[f] = -\mathcal{P} \circ \Theta \circ \mathcal{T}^{-1} \circ \Theta^*[f] = \Theta \circ \mathcal{T} \circ \mathcal{T}^{-1} \circ \Theta^*[f] = \Theta \circ \Theta^*[f] = f,$$

and we obtain the statement of the proposition for

$$h = \mathcal{P}^{-1}[f].$$

To find an explicit formula for $\mathcal{P}^{-1}$ we use explicit formulas for Θ and $\mathcal{T}^{-1}$, and obtain

$$\begin{aligned} \mathcal{P}^{-1}[f](x) &= \frac{1}{\pi x}\int_{-1}^{1} \sqrt{\frac{1-y^2}{1-1/x^2}} \cdot \frac{f(1/y)}{y(y-1/x)} dy \\ &= \frac{|x|}{\pi}\int_{-1}^{1} \sqrt{\frac{1-y^2}{x^2-1}} \left[\frac{1}{y} f\left(\frac{1}{y}\right)\right] \frac{dy}{xy-1}. \end{aligned} \tag{23}$$

$\square$

Remark. Following [Tr] we notice that solution of equation (19) is unique in $\mathcal{L}^2\left(I^c(1)\right)$, but is not unique in larger spaces. Namely, function

$$h(x) = \frac{1}{\sqrt{x^2-1}}$$

is a solution, and the only one in $\mathcal{L}^{2-}\left(I^c(1)\right)$ up to the linear dependence, of the homogeneous equation

$$\mathcal{P}[h] = 0.$$

□

5. Solvability of equation (16).

From the asymptotic expansions of $K_0(\zeta)$ (see [EMOT], [GR]) we obtain the following representations of the function $R(x,\lambda)$ for λ such that $\mathrm{Re}\lambda \in [\sigma_1, \sigma_2]$ with $\sigma_1 > \sigma_a$:

$$\begin{aligned} R(x,\lambda) &= -\frac{U}{x} + \lambda \log\left(\lambda|x|\right)\alpha(\lambda|x|) + \lambda\beta\left(\lambda|x|\right) + \gamma\left(\lambda|x|\right) \quad \text{for } |\lambda x| \le B, \\ R(x,\lambda) &= \lambda\delta\left(\lambda|x|\right)\frac{e^{-(\sigma+i\eta)|x|}}{\sqrt{|\lambda|\cdot|x|}} \quad \text{for } |\lambda x| > B, \end{aligned} \tag{24}$$

where $\alpha(\zeta)$, $\beta(\zeta)$, $\gamma(\zeta)$, and $\delta(\zeta)$ are bounded analytic functions on $\mathrm{Re}\zeta > \epsilon > 0$ and $B > 0$ is some constant.

Using representations (24) we introduce function $M(x,\lambda)$, analytic with respect to $\lambda \in \{\mathrm{Re}\lambda > \sigma_a\}$, uniquely defined by (24), and such that

$$R(x,\lambda) = -\frac{U}{x} + M(x,\lambda).$$

We consider then operators

$$\mathcal{M}_\lambda[f](x) = \int_{|y|>1} \chi_A(x) M(x-y,\lambda) f(y)dy,$$

and

$$\mathcal{R}_\lambda = \pi U \cdot \mathcal{P} + \mathcal{M}_\lambda.$$

In the next proposition we prove compactness of the operator $\frac{1}{\pi U}\mathcal{M}_\lambda \circ \mathcal{P}^{-1}$ on $\mathcal{L}^2\left(I^c(1)\right)$.

Proposition 5.1. *For any fixed $\lambda \in \mathbb{C}$ operator $\mathcal{N}_\lambda = \frac{1}{\pi U}\mathcal{M}_\lambda \circ \mathcal{P}^{-1}$ is compact on $\mathcal{L}^2\left(I^c(1)\right)$, and therefore operator*

$$\mathcal{G}_\lambda = \mathcal{R}_\lambda \circ \mathcal{P}^{-1} = \left(\pi U \cdot \mathcal{P} + \mathcal{M}_\lambda\right) \circ \mathcal{P}^{-1} = \pi U\left(\mathcal{I} + \mathcal{N}_\lambda\right), \tag{25}$$

where $\mathcal{I}$ is the identity operator, is a Fredholm operator on $\mathcal{L}^2\left(I^c(1)\right) = L^2\left(I^c(1)\right)$. In addition, kernel $N(x,y,\lambda)$ of the operator $\mathcal{N}_\lambda$ admits estimate

$$\int_{\mathbb{R}^2} |N(x,y,\lambda)|^2 dxdy < C|\lambda\log\lambda|^2 \tag{26}$$

with constant C independent of λ.

Proof. Using formula (23) for $\mathcal{P}^{-1}$, we obtain

$$\mathcal{N}_\lambda[g](x) = \mathcal{M}_\lambda\left[\frac{|x|}{\pi^2 U}\int_{-1}^{1}\sqrt{\frac{1-u^2}{x^2-1}}\left[\frac{1}{u}g\left(\frac{1}{u}\right)\right]\frac{du}{xu-1}\right]$$

$$= \mathcal{M}_\lambda \left[\frac{|x|}{\pi^2 U} \int_{|y|>1} \sqrt{\frac{1-\frac{1}{y^2}}{x^2-1}} y^2 g(y) \frac{dy}{y^2(x-y)} \right]$$

$$= \frac{\chi_A(x)}{\pi^2 U} \int_{|u|>1} M(x-u,\lambda) du \int_{|y|>1} \frac{|u|\sqrt{y^2-1}}{|y|\sqrt{u^2-1}} g(y) \frac{dy}{(u-y)}$$

$$= \frac{\chi_A(x)}{\pi^2 U} \int_{|y|>1} g(y) dy \int_{|u|>1} M(x-u,\lambda) \frac{|u|\sqrt{y^2-1}}{|y|\sqrt{u^2-1}} \frac{du}{(u-y)} = \int_{|y|>1} N(x,y,\lambda) g(y) dy$$

with kernel

$$N(x,y,\lambda) = \frac{\chi_A(x)}{\pi^2 U} \int_{|u|>1} M(x-u,\lambda) \frac{|u|\sqrt{y^2-1}}{|y|\sqrt{u^2-1}} \frac{du}{(u-y)}.$$

To prove compactness of operator $\mathcal{N}_\lambda$ we use representation

$$N(x,y,\lambda) = \frac{1}{\pi^2 U} \left[N_1(x,y,\lambda) + N_2(x,y,\lambda) \right],$$

with

$$N_1(x,y,\lambda) = \frac{\chi_A(x)}{|y|} \int_{|u|>1} M(x-u,\lambda)|u| \frac{du}{(u-y)},$$

and

$$N_2(x,y,\lambda) = \frac{\chi_A(x)}{|y|} \int_{|u|>1} M(x-u,\lambda) \frac{|u|\left(\sqrt{y^2-1}-\sqrt{u^2-1}\right)}{\sqrt{u^2-1}} \frac{du}{(u-y)}$$

$$= -\frac{\chi_A(x)}{|y|} \int_{|u|>1} M(x-u,\lambda) \frac{|u|\,(y+u)\,du}{\left(\sqrt{y^2-1}+\sqrt{u^2-1}\right)\sqrt{u^2-1}},$$

and prove Hilbert-Schmidt property (cf.[L]) of kernels $N_1(x,y,\lambda)$ and $N_2(x,y,\lambda)$.

For $N_1(x,y,\lambda)$ we notice that for fixed x satisfying $1<|x|<A$

$$\int_{|u|>1} M(x-u,\lambda)|u| \frac{du}{(u-y)}$$

is a multiple of the Hilbert transform of an $L^2\left(I^c(1)\right)$ - function $M(x-u,\lambda)|u|$ with

$$\|M(x-u,\lambda)|u|\|_{L^2_u} < \infty.$$

Therefore we have

$$\begin{aligned} &\int_{1<|x|<A} dx \int_{|y|>1} dy \left|N_1(x,y,\lambda)\right|^2 \\ &= \int_{1<|x|<A} dx \int_{|y|>1} dy \left| \frac{1}{|y|} \int_{|u|>1} M(x-u,\lambda)|u| \frac{du}{(u-y)} \right|^2 \\ &< C \int_{1<|x|<A} dx \, \|M(x-u,\lambda)|u|\|^2_{L^2_u} < \infty. \end{aligned} \tag{27}$$

For $N_2(x,y,\lambda)$ we have

$$\begin{aligned}
&\int_{1<|x|<A} dx \int_{|y|>1} dy\, |N_2(x,y,\lambda)|^2 \\
&= \int_{1<|x|<A} dx \int_{|y|>1} \frac{dy}{|y|^2} \left| \int_{|u|>1} M(x-u,\lambda) \frac{|u|\,(y+u)\,du}{\left(\sqrt{y^2-1}+\sqrt{u^2-1}\right)\sqrt{u^2-1}} \right|^2 \\
&\le 2 \int_{1<|x|<A} dx \int_{|y|>1} \frac{dy}{|y|^2} \left| \int_0^\infty M(x-\sqrt{t^2+1},\lambda) \frac{\left(y+\sqrt{t^2+1}\right)dt}{\left(\sqrt{y^2-1}+t\right)} \right|^2 \\
&+2 \int_{1<|x|<A} dx \int_{|y|>1} \frac{dy}{|y|^2} \left| \int_0^\infty M(x+\sqrt{t^2+1},\lambda) \frac{\left(y-\sqrt{t^2+1}\right)dt}{\left(\sqrt{y^2-1}+t\right)} \right|^2,
\end{aligned} \tag{28}$$

where we changed variable to $t=\sqrt{u^2-1}$.

Both integrals of the right hand side of (28) are estimated analogously, therefore we will present an estimate of the first of them only.

For $1<|x|<A$ and $|y|>2$ we have inequality

$$\left| \frac{y+\sqrt{t^2+1}}{\sqrt{y^2-1}+t} \right| < C \tag{29}$$

for some C independent of y, and therefore, using representations (24), we obtain

$$\begin{aligned}
&\int_{1<|x|<A} dx \int_{|y|>2} \frac{dy}{|y|^2} \left| \int_0^\infty M(x-\sqrt{t^2+1},\lambda) \frac{\left(y+\sqrt{t^2+1}\right)dt}{\left(\sqrt{y^2-1}+t\right)} \right|^2 \\
&\le C^2 \int_{1<|x|<A} dx \int_{|y|>2} \frac{dy}{|y|^2} \left| \int_0^\infty M(x-\sqrt{t^2+1},\lambda)dt \right|^2 < \infty.
\end{aligned} \tag{30}$$

For $1<|x|<A$, $1<|y|<2$, and $t>A+B$ we again use inequality (29) and obtain

$$\begin{aligned}
&\int_{1<|x|<A} dx \int_{1<|y|<2} \frac{dy}{|y|^2} \left| \int_{A+B}^\infty M(x-\sqrt{t^2+1},\lambda) \frac{\left(y+\sqrt{t^2+1}\right)dt}{\left(\sqrt{y^2-1}+t\right)} \right|^2 \\
&\le C^2 \int_{1<|x|<A} dx \int_{|y|>2} \frac{dy}{|y|^2} \left| \int_0^\infty M(x-\sqrt{t^2+1},\lambda)dt \right|^2 < \infty.
\end{aligned} \tag{31}$$

For $1<|x|<A$, $1<|y|<2$, and $t<A+B$ we have

$$\begin{aligned}
&\int_{1<|x|<A} dx \int_{1<|y|<2} \frac{dy}{|y|^2} \left| \int_0^{A+B} M(x-\sqrt{t^2+1},\lambda) \frac{\left(y+\sqrt{t^2+1}\right)dt}{\left(\sqrt{y^2-1}+t\right)} \right|^2 \\
&< C \int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_0^{A+B} \frac{M(x-\sqrt{t^2+1},\lambda)dt}{\left(\sqrt{y^2-1}+t\right)} \right|^2 \\
&< C|\lambda \log \lambda|^2 \int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_0^{A+B} \frac{dt}{\left(\sqrt{y^2-1}+t\right)} \right|^2
\end{aligned}$$

$$+C|\lambda|^2\int_{1<|x|<A}dx\int_{1<|y|<2}dy\left|\int_0^{A+B}\frac{\log|x-\sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)}dt\right|^2$$

$$< C|\lambda|^2\left(|\log\lambda|^2+\int_{1<|x|<A}dx\int_{1<|y|<2}dy\left|\int_0^{A+B}\frac{\log|x-\sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)}dt\right|^2\right),$$

where we used representation

$$M(x-\sqrt{t^2+1},\lambda)=\lambda\left(\log\lambda+\log|x-\sqrt{t^2+1}|\right)\alpha(\lambda|x-\sqrt{t^2+1}|)$$

$$+\lambda\beta(\lambda|x-\sqrt{t^2+1}|)+\gamma(\lambda|x-\sqrt{t^2+1}|)$$

for $1<|x|<A$ and $0<t<A+B$, which is a corollary of (24).

To estimate the last integral we represent it as

$$\int_{1<|x|<A}dx\int_{1<|y|<2}dy\left|\int_0^{A+B}\frac{\log|x-\sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)}dt\right|^2$$

$$=\int_{1<|x|<A}dx\int_{1<|y|<2}dy\left|\int_{S(x,y)}\frac{\log|x-\sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)}dt\right|^2$$

$$+\int_{1<|x|<A}dx\int_{1<|y|<2}dy\left|\int_{\{[0,A+B]\setminus S(x,y)\}}\frac{\log|x-\sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)}dt\right|^2$$

where $S(x,y)=\left\{t:\left|x-\sqrt{t^2+1}\right|\geq\frac{1}{2}(x-1)\sqrt{y^2-1}\right\}$.

Then for $S(x,y)$ we have

$$\int_{1<|x|<A}dx\int_{1<|y|<2}dy\left|\int_{S(x,y)}\frac{\log|x-\sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)}dt\right|^2$$

$$\leq C\int_{1<|x|<A}dx\int_{1<|y|<2}dy\left|\int_0^{A+B}\frac{\log|x-1|+\log\left(\sqrt{y^2-1}\right)}{\left(\sqrt{y^2-1}+t\right)}dt\right|^2$$

$$\leq C\int_{1<|x|<A}dx\int_{1<|y|<2}dy\log^2\sqrt{y^2-1}\cdot\left(\log|x-1|+\log\sqrt{y^2-1}\right)^2<C<\infty.$$

For $t\in[0,A+B]\setminus S(x,y)$ we have

$$1+\frac{t^2}{2}\geq\sqrt{t^2+1}>x-\frac{1}{2}(x-1)\sqrt{y^2-1},$$

and therefore

$$\frac{t^2}{2}\geq(x-1)\left[1-\frac{1}{2}\sqrt{y^2-1}\right],$$

or

$$t\geq C\sqrt{x-1}.$$

Using the last inequality we obtain

$$|dt|\leq\left|\frac{\sqrt{t^2+1}}{t}du\right|\leq C\left|\frac{1}{\sqrt{x-1}}du\right|,$$

and switching to variable $u=\sqrt{t^2+1}$ for $[0,A+B]\setminus S(x,y)$, we obtain

$$\int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_{\{[0,A+B]\setminus S(x,y)\}} \frac{\log|x-\sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)} dt \right|^2$$

$$\leq C \int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_{x-\frac{1}{2}(x-1)\sqrt{y^2-1}}^{x+\frac{1}{2}(x-1)\sqrt{y^2-1}} \frac{\log|u-x|}{\sqrt{y^2-1}\cdot\sqrt{x-1}} du \right|^2$$

$$\leq C \int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \frac{(x-1)\sqrt{y^2-1}\cdot\left(\log(x-1)+\log\sqrt{y^2-1}\right)}{\sqrt{y^2-1}\cdot\sqrt{x-1}} \right|^2 < C < \infty.$$

Combining the last two estimates above we obtain

$$\int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_0^{A+B} \frac{\log|x-\sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)} dt \right|^2 < C < \infty,$$

and therefore

$$\int_{1<|x|<A} dx \int_{1<|y|<2} \frac{dy}{|y|^2} \left| \int_0^{A+B} M(x-\sqrt{t^2+1},\lambda) \frac{\left(y+\sqrt{t^2+1}\right) dt}{\left(\sqrt{y^2-1}+t\right)} \right|^2 < C|\lambda\log\lambda|^2. \tag{32}$$

To prove estimate (26) we use the following lemma.

Lemma 5.2. *The following estimates hold for $1<|x|<A$ and $Re\lambda\in[\sigma_1,\sigma_2]$*

$$\begin{gathered}\int_{\mathbb{R}} |M(x-u,\lambda)|^2 u^2 du < C|\lambda|^{1+\epsilon} \text{ for arbitrary } \epsilon>0, \\ \left| \int_0^\infty M(x-\sqrt{t^2+1},\lambda) dt \right| < C\sqrt{|\lambda|}.\end{gathered} \tag{33}$$

Proof. Using representation (24) for $1<|x|<A$ and $|\lambda(x-u)|\leq B$ we obtain

$$M(x-u,\lambda)\cdot u = \left[\lambda\log(\lambda|x-u|)\,\alpha(\lambda|x-u|) + \lambda\beta(\lambda|x-u|) + \gamma(\lambda|x-u|)\right] u,$$

and therefore

$$\int_{|x-u|\leq B/|\lambda|} |M(x-u,\lambda)|^2 u^2 du$$

$$< C \int_{|x-u|\leq B/|\lambda|} \Big[|\lambda|^2 \left(|\log\lambda|^2 + \log^2|x-u| \right) |\alpha(\lambda|x-u|)|^2$$

$$+ |\lambda|^2 |\beta(\lambda|x-u|)|^2 + |\gamma(\lambda|x-u|)|^2 \Big] u^2 du < C|\lambda||\log\lambda|^2.$$

For $1<|x|<A$ and $|\lambda(x-u)|\geq B$ from (24) we have

$$M(x-u,\lambda) = \lambda\delta(\lambda|x-u|) \frac{e^{-(\sigma+i\eta)|x-u|}}{\sqrt{|\lambda|\cdot|x-u|}},$$

and therefore

$$\int_{|x-u|\geq B/|\lambda|} |M(x-u,\lambda)|^2 u^2 dx$$

$$< C \int_{\mathbb{R}} |\lambda|^{1+\epsilon} |\delta(\lambda|x-u|)|^2 \frac{e^{-2\sigma|x-u|} u^2 du}{(|\lambda(x-u)|)^\epsilon |x-u|^{1-\epsilon}} < C|\lambda|^{1+\epsilon}$$

for any $\epsilon>0$.

Combining the estimates above we obtain the first estimate from (33).

For the second integral in (33) we use representation (24), and obtain for $1 < |x| < A$ and $|\lambda(x - \sqrt{t^2+1})| \leq B$

$$\left|\int_{|x-\sqrt{t^2+1}|\leq B/|\lambda|} M(x - \sqrt{t^2+1}, \lambda)dt\right|$$
$$\leq \int_{|x-\sqrt{t^2+1}|\leq B/|\lambda|} \left|\lambda \log\left(\lambda|x - \sqrt{t^2+1}|\right) \alpha(\lambda|x - \sqrt{t^2+1}|)\right.$$
$$\left. +\lambda\beta\left(\lambda|x - \sqrt{t^2+1}|\right) + \gamma\left(\lambda|x - \sqrt{t^2+1}|\right)\right| dt \leq C|\log\lambda|,$$

where we used the fact that the length of the interval of integration is bounded by $\dfrac{C}{|\lambda|}$ for some $C > 0$.

For $1 < |x| < A$ and $|\lambda(x - \sqrt{t^2+1})| > B$ using representation (24) we obtain

$$\left|\int_{|x-\sqrt{t^2+1}|>B/|\lambda|} M(x - \sqrt{t^2+1}, \lambda)dt\right|$$
$$< C\int_{\mathbb{R}} \lambda\delta\left(\lambda|x - \sqrt{t^2+1}|\right) \frac{e^{-(\sigma+i\eta)|x-\sqrt{t^2+1}|}dt}{\sqrt{|\lambda|\cdot|x - \sqrt{t^2+1}|}} < C\sqrt{|\lambda|}.$$

Combining the two estimates above we obtain the second estimate of (33). □

Using now estimates (33) from the lemma above in estimates (27), (30), and (31) and combining them with estimate (32) we obtain estimate (26) of Proposition 5.1. □

Proposition 5.1 allows us to reduce the question of solvability of (16) to the solvability of corresponding equation for $\mathcal{G}_\lambda$. Namely, calling those λ for which operator $\mathcal{G}_\lambda$ is not invertible by *characteristic values of* $\mathcal{G}_\lambda$, we have

Proposition 5.3. *If λ_0 is not a characteristic value of $\mathcal{G}_\lambda$, then for arbitrary function $f \in \mathcal{L}^2(I^c(1))$ and $\lambda = \lambda_0$ there exists a solution h of equation (16) such that $h \in \mathcal{L}^p(I^c(1))$ for any $p < \dfrac{4}{3}$.*

Proof. Considering a solution of

$$\mathcal{G}_\lambda[g] = \mathcal{R}_\lambda \circ \mathcal{P}^{-1}[g] = f$$

we define $h = \mathcal{P}^{-1}[g]$, which satisfies equation (16) and belongs to $\mathcal{L}^p(I^c(1))$ for any $p < \dfrac{4}{3}$ according to Proposition 4.1.

□

6. The resolvent of operator $\mathcal{G}_\lambda$.

In this section we construct the resolvent of the operator $\mathcal{G}_\lambda$ and show that it is a Fredholm operator also analytically depending on $\lambda \in \{\mathrm{Re}\lambda > \sigma_1\}$.

Let $\mathcal{T} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be an integral operator with kernel $T(x,y)$ satisfying Hilbert-Schmidt condition. Following [C], we consider for operator $\mathcal{T}$ Hilbert's modification of the original Fredholm's determinants:

$$\mathcal{D}_{T,m}(t_1,\ldots,t_m) = \begin{vmatrix} 0 & T(t_1,t_2) & \cdots & T(t_1,t_m) \\ T(t_2,t_1) & 0 & \cdots & T(t_2,t_m) \\ \vdots & & & \vdots \\ T(t_m,t_1) & \cdots & T(t_m,t_{m-1}) & 0 \end{vmatrix},$$

$$\mathcal{D}_T = 1 + \sum_{m=1}^{\infty} \delta_m = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathcal{D}_{T,m}\left(t_1, \ldots, t_m\right) dt_1 \cdots dt_m, \tag{34}$$

$$\mathcal{D}_{T,m}\left(\begin{matrix} x \\ y \end{matrix} \; t_1, \ldots, t_m \right) = \left| \begin{matrix} T(x,y) & T(x,t_1) & \cdots & T(x,t_m) \\ T(t_1,y) & 0 & \cdots & T(t_1,t_m) \\ \vdots & & & \vdots \\ T(t_m,y) & \cdots & T(t_m,t_{m-1}) & 0 \end{matrix} \right|,$$

and

$$\begin{aligned} \mathcal{D}_T \left(\begin{matrix} x \\ y \end{matrix} \right) &= T(x,y) + \sum_{m=1}^{\infty} \delta_m \left(\begin{matrix} x \\ y \end{matrix} \right) \\ &= T(x,y) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathcal{D}_{T,m} \left(\begin{matrix} x \\ y \end{matrix} \; t_1, \ldots, t_m \right) dt_1 \cdots dt_m. \end{aligned} \tag{35}$$

We start with the following proposition, which summarizes the results from [C] (cf. also [M]), that will be used in the construction of the resolvent of $\mathcal{G}_\lambda$.

Proposition 6.1. *([C]) Let function $T(x,y) : \mathbb{R}^2 \to \mathbb{C}$ satisfy Hilbert-Schmidt condition*

$$\|T\|^2 = \int_{\mathbb{R}^2} |T(x,y)|^2 \, dxdy < \infty.$$

Then function $\mathcal{D}_T \left(\begin{matrix} x \\ y \end{matrix} \right) \in L^2(\mathbb{R}^2)$ is well defined, and the following estimates hold:

$$|\delta_m| \le \left(\frac{e}{m} \right)^{m/2} \|T\|^m, \; |\mathcal{D}_T| \le e^{\frac{\|T\|^2}{2}}, \tag{36}$$

$$\left| \mathcal{D}_T \left(\begin{matrix} x \\ y \end{matrix} \right) \right| \le e^{\frac{\|T\|^2}{2}} \left(|T(x,y)| + \sqrt{e}\alpha(x)\beta(y) \right), \tag{37}$$

where

$$\alpha^2(x) = \int_{\mathbb{R}} |T(x,t)|^2 \, dt, \quad \beta^2(y) = \int_{\mathbb{R}} |T(t,y)|^2 \, dt.$$

If $\mathcal{D}_T \neq 0$ then kernel

$$H(x,y) = [\mathcal{D}_T]^{-1} \cdot \mathcal{D}_T \left(\begin{matrix} x \\ y \end{matrix} \right) \tag{38}$$

defines the resolvent of operator $\mathcal{I} - \mathcal{T}$, i.e. it satisfies the following equations

$$\begin{aligned} H(x,y) + \int_{\mathbb{R}} T(x,t) \cdot H(t,y) dt = T(x,y), \\ H(x,y) + \int_{\mathbb{R}} T(t,y) \cdot H(x,t) dt = T(x,y), \end{aligned} \tag{39}$$

and therefore operator $\mathcal{I} - \mathcal{H}$ is the inverse of operator $\mathcal{I} + \mathcal{T}$.

□

Using Proposition 6.1, we construct the resolvent of operator $\mathcal{G}_\lambda = \pi U \left(\mathcal{I} + \mathcal{N}_\lambda \right)$, defined in (25), and prove the estimate that will be necessary in the proof of Theorem 1.

Proposition 6.2. *The set of characteristic values of operator $\mathcal{G}_\lambda$ coincides with the set*

$$E(\mathcal{G}) = \{\lambda \in \mathbb{C}:\ Re\lambda > \sigma_1,\ \mathcal{D}_{N_\lambda} = 0\}$$

and consists of at most countably many isolated points.

For $\lambda \notin E(\mathcal{G})$ there exists an operator $\mathcal{H}_\lambda$ with kernel $H(x,y,\lambda)$ satisfying the Hilbert-Schmidt condition and such that operator $\mathcal{I}-\mathcal{H}_\lambda$ is the inverse of operator $\mathcal{I}+\mathcal{N}_\lambda$, and therefore operator $\dfrac{1}{\pi U}(\mathcal{I}-\mathcal{H}_\lambda)$ is the inverse of operator $\mathcal{G}_\lambda$.

If function $\mathcal{D}_N(\lambda) = \mathcal{D}_{N_\lambda}$ has no zeros in a strip $\{\lambda:\ \sigma_1 < Re\lambda < \sigma_2\}$, then operator $\mathcal{H}_\lambda$ admits estimate

$$||\mathcal{H}_\lambda\| < \exp\left\{e^{|\eta|}\cdot(1+|\eta|)^{2+\epsilon}\right\} \tag{40}$$

for $\lambda \in \{\sigma_1+\gamma < Re\lambda < \sigma_2-\gamma\}$ and arbitrary $\epsilon > 0$.

Proof. Applying Proposition 6.1 to operator $\mathcal{N}_\lambda$ we obtain the existence of functions

$$\mathcal{D}_N(\lambda) = \mathcal{D}_{N_\lambda} \tag{41}$$

and

$$\mathcal{D}_N\left(\begin{array}{c} x \\ y \end{array}\middle|\lambda\right) = \mathcal{D}_{N_\lambda}\left(\begin{array}{c} x \\ y \end{array}\right)$$

such that for any fixed λ, satisfying $\mathcal{D}_N(\lambda) \neq 0$, kernel

$$H(x,y,\lambda) = [\mathcal{D}_N(\lambda)]^{-1}\cdot\mathcal{D}_N\left(\begin{array}{c} x \\ y \end{array}\middle|\lambda\right) \in L^2(\mathbb{R}^2),$$

and operator $\mathcal{I}-\mathcal{H}_\lambda$ is the inverse of operator $\mathcal{I}+\mathcal{N}_\lambda$.

Terms of the series (34) for $\mathcal{N}_\lambda$ analytically depend on λ, and according to estimates (36) this series converges uniformly with respect to λ on compact subsets of $\{\lambda \in \mathbb{C}:\ \mathrm{Re}\lambda > \sigma_1\}$. Therefore, $\mathcal{D}_N(\lambda)$ is an analytic function on $\{\lambda \in \mathbb{C}:\ \mathrm{Re}\lambda > \sigma_1\}$, and the set $E(\mathcal{G})$ consists of at most countably many isolated points.

Analyticity of $\mathcal{I}-\mathcal{H}_\lambda$ with respect to λ on

$$\{\lambda \in \mathbb{C}:\ \mathrm{Re}\lambda > \sigma_1\}\setminus E(\mathcal{G})$$

follows from the Theorem VI.14 in [RS]. It is proved by approximation of the kernel by degenerate kernels and by the argument that can be traced back to at least [M].

To prove estimate (40) we use the well known estimate ([L])

$$\|T\|^2 \leq \int_{\mathbb{R}} |T(x,y)|^2 dxdy$$

for integral operators. Using this estimate, estimates (37) and (26) we obtain

$$\left\|\mathcal{D}_N\left(\begin{array}{c} x \\ y \end{array}\middle|\lambda\right)\right\| < \exp\left\{C(1+|\eta|)^2\cdot\log^2|\eta|\right\}(1+|\eta|)^4\cdot\log^4|\eta|.$$

To estimate function $[\mathcal{D}_N(\lambda)]^{-1}$ for $\lambda \in \{\sigma_1+\gamma < \mathrm{Re}\lambda < \sigma_2-\gamma\}$ we use the following lemma.

Lemma 6.3. *If function $\mathcal{D}_N(\lambda) = \mathcal{D}_{N_\lambda}$ has no zeros in the strip $\{\lambda:\ \sigma_1 < Re\lambda < \sigma_2\}$, then estimate*

$$|1/\mathcal{D}_N(\lambda)| < \exp\left\{e^{|\eta|}\cdot(1+|\eta|)^{2+\epsilon}\right\} \tag{42}$$

holds for $\lambda \in \{\sigma_1+\gamma < Re\lambda < \sigma_2-\gamma\}$ with fixed $\gamma > 0$ and arbitrary $\epsilon > 0$.

Proof. We consider a biholomorphic map

$$\Psi : \ \{\lambda : \ \sigma_1 < \mathrm{Re}\lambda < \sigma_2\} \to \mathbb{D}(1) = \{z \in \mathbb{C} : |z| < 1\},$$

defined by the formula

$$\Psi(\lambda) = \frac{e^{i(\lambda-\sigma_1)\frac{\pi}{\sigma_2-\sigma_1}} - i}{e^{i(\lambda-\sigma_1)\frac{\pi}{\sigma_2-\sigma_1}} + i}.$$

Denoting

$$w = u + iv = e^{i(\lambda-\sigma_1)\frac{\pi}{\sigma_2-\sigma_1}},$$

we obtain for the circle $C(r) = \{z : \ |z| = r\}$

$$\begin{aligned}\Psi^{-1}\left(C(r)\right) &= \left\{\sigma + i\eta : \ \left|e^{i(\lambda-\sigma_1)\frac{\pi}{\sigma_2-\sigma_1}} - i\right| = r\left|e^{i(\lambda-\sigma_1)\frac{\pi}{\sigma_2-\sigma_1}} + i\right|\right\} \\ &= \left\{u + iv : \ \left(u^2 + v^2 - 2v + 1\right) = r^2\left(u^2 + v^2 + 2v + 1\right)\right\} \\ &= \left\{u + iv : \ u^2 + \left(v - \frac{1+r^2}{1-r^2}\right)^2 = \frac{4r^2}{(1-r^2)^2}\right\}.\end{aligned}$$

Introducing coordinates

$$t = \mathrm{Re}\frac{\pi(\lambda-\sigma_1)}{\sigma_2-\sigma_1}, \ \ s = \mathrm{Im}\frac{\pi(\lambda-\sigma_1)}{\sigma_2-\sigma_1},$$

such that

$$w = u + iv = e^{i(\lambda-\sigma_1)\frac{\pi}{\sigma_2-\sigma_1}} = e^{it-s} = e^{-s}\left(\cos t + \sin t\right),$$

we can rewrite the last condition as a quadratic equation with respect to e^{-s} for fixed t

$$\left(e^{-s} - \sin t\frac{1+r^2}{1-r^2}\right)^2 + \cos^2 t\left(\frac{1+r^2}{1-r^2}\right)^2 - \frac{4r^2}{(1-r^2)^2} = 0.$$

Solving equation above we obtain

$$e^{-s} = \sin t\frac{1+r^2}{1-r^2} \pm \sqrt{\frac{4r^2}{(1-r^2)^2} - \cos^2 t\left(\frac{1+r^2}{1-r^2}\right)^2}$$

with solutions existing for t such that

$$|\cos t| \leq \frac{2r}{1-r^2}\frac{1-r^2}{1+r^2} = \frac{2r}{1+r^2}.$$

The maximal value for e^{-s} is achieved at $t = \frac{\pi}{2}$ and it is

$$e^{-s} = \frac{1+r^2}{1-r^2} + \frac{2r}{1-r^2} = \frac{1+r^2+2r}{1-r^2} = \frac{(1+r)^2}{1-r^2} = \frac{1+r}{1-r}.$$

Therefore the maximal value for $|s|$ is achieved at $t = \frac{\pi}{2}$, is equal to $|s| = \log\left(\frac{1+r}{1-r}\right)$, and for $r = 1 - \delta$ we have the maximal value

$$\max|s| = \log\left(\frac{1+r}{1-r}\right) = -\log\delta + \log\left(2-\delta\right). \tag{43}$$

Since function $\mathcal{D}_N\left(\lambda\right)$ has no zeros in $\{\lambda : \ \sigma_1 < \mathrm{Re}\lambda < \sigma_2\}$ we can consider analytic function $\log\left(\mathcal{D}_N\left(\lambda\right)\right)$ in this strip, and using estimates (36) and (26), and equality (43), we obtain the following estimate for $z = (1-\delta)e^{i\theta}$

$$\begin{aligned}\log\left|\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right| &\leq \frac{\left\|N_{\Psi^{-1}(z)}\right\|^2}{2} \leq C\left|\Psi^{-1}(z)\cdot\log\left(\Psi^{-1}(z)\right)\right|^2 \\ &\leq C|\log\delta\cdot\log\left(\log\delta\right)|^2.\end{aligned}$$

Using then the Borel-Caratheodory inequality ([Ti1], [Boa]) on disks with radii

$$1-2\delta = r < R = 1-\delta,$$

we obtain

$$\left|\log\left(\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right)\right|_{\{|z|=1-2\delta\}}$$

$$\leq \frac{2-4\delta}{\delta}\max_{|z|=R}\mathrm{Re}\left\{\log\left(\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right)\right\} + \frac{1-\delta+1-2\delta}{\delta}\left|\log\left(\mathcal{D}_N\left(\Psi^{-1}(0)\right)\right)\right|$$

$$< \frac{C}{\delta}\log^2\delta\cdot\log^2(\log\delta),$$

or

$$-\frac{C}{\delta}\log^2\delta\cdot\log^2(\log\delta) < \mathrm{Re}\left\{\log\left(\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right)\right\}\Big|_{\{|z|=1-2\delta\}} < \frac{C}{\delta}\log^2\delta\cdot\log^2(\log\delta).$$

From the last estimate we obtain an estimate for the function $\left|1/\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right|$ in the disk $\mathbb{D}(1-2\delta)$:

$$\left|1/\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right|\Big|_{\{|z|\leq 1-2\delta\}} < \exp\left\{\frac{|\log\delta|^{2+\epsilon}}{\delta}\right\} \tag{44}$$

for arbitrary $\epsilon > 0$.

For a fixed $t \in (0,\pi)$ and arbitrary s we have that $t+is \in \Psi^{-1}(\mathbb{D}(r))$ with $r = 1-2\delta$ if

$$e^{|s|} \leq \sin t\cdot\frac{1+r^2}{1-r^2} + \sqrt{\frac{4r^2}{(1-r^2)^2} - \cos^2 t\cdot\left(\frac{1+r^2}{1-r^2}\right)^2}$$

$$= \sin t\cdot\frac{2-4\delta+4\delta^2}{2\delta(2-2\delta)} + \frac{\sqrt{4(1-2\delta)^2-\cos^2 t\cdot(2-4\delta+4\delta^2)^2}}{2\delta(2-2\delta)},$$

and therefore for any interval $[\gamma',\pi-\gamma']$ there exist constants $C_1,\ C_2$ such that conditions

$$t\in\left[\gamma',\pi-\gamma'\right],\quad \frac{C_1}{\delta} < e^{|s|} < \frac{C_2}{\delta}$$

imply that $t+is \in \Psi^{-1}(\mathbb{D}(1-2\delta))$.

Using then estimate (44) we obtain for λ with $\mathrm{Re}\lambda \in \left[\sigma_1 + \frac{\gamma'(\sigma_2-\sigma_1)}{\pi}, \sigma_2 - \frac{(\pi-\gamma')(\sigma_2-\sigma_1)}{\pi}\right]$ the estimate

$$|1/\mathcal{D}_N(\lambda)| < \exp\left\{e^{|s|}\cdot(1+|s|)^{2+\epsilon}\right\}$$

for arbitrary $\epsilon > 0$, which leads to estimate (42). □

Combining now estimate for $\left\|\mathcal{D}_N\left(\begin{array}{c} x\\ y\end{array}\Big|\lambda\right)\right\|$ with (42) we obtain estimate (40). □

7. Proof of Theorem 1.

Before proving Theorem 1 we will prove two lemmas, that will be used in the proof of this theorem.

In order to assure applicability of Proposition 5.3 to f_a, defined in (17), we have to prove that

$$f_a \in \mathcal{L}^2\left(I^c(1)\right)$$

for $\widehat{w_a}$ satisfying (4). In the lemma below we prove the necessary property of f_a.

Lemma 7.1. *If $\widehat{w_a}$ satisfies condition (4) then $f_a(x,\lambda)$ defined by the formula (17) is a function in $\mathcal{L}^2\left(I^c(1)\right)$ for any fixed λ, which satisfies the estimate*

$$\|f_a(\cdot,\sigma+i\eta)\|_{\mathcal{L}^2(I^c(1))} < C\exp\left\{-e^{|\eta|}\cdot(1+|\eta|)^{2+\epsilon}\right\} \tag{45}$$

with some $\epsilon>0$ for $\sigma\in[\sigma_1,\sigma_2]$.

Proof. For a fixed $\lambda=\sigma+i\eta$ with $\sigma\in[\sigma_1,\sigma_2]$ we choose $B>1$, and using second representation from (24) of $R(x,\lambda)$ for $|\lambda x|>B$, obtain an estimate

$$|R(x-y,\lambda)| < C\frac{|\lambda|^{1/2}e^{-\lambda|x-y|}}{\sqrt{|x-y|}}.$$

Using then condition (4), we have

$$\begin{aligned}
&\left(\int_{|x|>B/|\lambda|}|f_a(x,\lambda)|^2\,dx\right)^{1/2} \qquad (46)\\
&=\frac{1}{\pi^2}\left|\int_{|x|>B/|\lambda|}\left|\int_{-1}^{1}e^{-d(\lambda)y}R(x-y,\lambda)\widehat{w_a}(y,\lambda)dy\right|^2dx\right|^{1/2}\\
&< C|\lambda|^{1/2}\left(\int_{|x|>B/|\lambda|}\left(\int_{-1}^{1}e^{-\sigma|x|}\,|\widehat{w_a}(y,\lambda)|\,dy\right)^2dx\right)^{1/2}\\
&< C|\lambda|^{1/2}\int_{-1}^{1}|\widehat{w_a}(y,\lambda)|\,dy < C|\lambda|^{1/2}\int_{-1}^{1}|\widehat{w_a}(y,\lambda)|^2\,dy\\
&< C\exp\left\{-e^{|\eta|}\cdot(1+|\eta|)^{2+\epsilon}\right\}.
\end{aligned}$$

For $|\lambda x|<B$ we use the first representation from (24) for $R(x-y,\lambda)$. Since the Hilbert transform is a bounded linear operator from L^q into L^q (see [Ti2], [Tr]), and kernels $\alpha\left(\lambda(x-y)\right)$, $\beta\left(\lambda(x-y)\right)$, and $\gamma\left(\lambda(x-y)\right)$ from (24) are bounded, we obtain

$$\begin{aligned}
\left(\int_{|x|<|B/\lambda|}|f_a(x,\lambda)|^2\,dx\right)^{1/2} &< C\,|\lambda\log\lambda|\cdot\|\widehat{w_a}(y,\lambda)\|_{L^2(I(1))} \qquad (47)\\
&< C\exp\left\{-e^{|\eta|}\cdot(1+|\eta|)^{2+\epsilon}\right\},
\end{aligned}$$

where in the last inequality we used condition (4).

Combining estimates (46) and (47) we obtain (45). □

Lemma 7.2. *If a function $h(y,\lambda)$ satisfies estimate*

$$\int_{-\infty}^{\infty}e^{-\sigma_1\cdot|y|}|h(y,\sigma+i\eta)|dy < \frac{C}{(1+|\eta|)^{\frac{5}{2}+\epsilon}} \tag{48}$$

for some $\epsilon>0$ and $\sigma_1<Re\lambda<\sigma_2$, then function

$$\xi(x,z,\lambda)=e^{d(\sigma+i\eta)x}\left[\int_{-\infty}^{\infty}K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right)h(y,\sigma+i\eta)dy\right]\in L^1_\eta(\mathbb{R}) \tag{49}$$

for $\sigma\in[\sigma_1,\sigma_2]$, and satisfies conditions

$$\frac{\partial^2\xi(x,z,\sigma+i\eta)}{\partial x^2},\ \frac{\partial^2\xi(x,z,\sigma+i\eta)}{\partial z^2},\ |\eta|^2\xi(x,z,\sigma+i\eta),\ |\eta|\frac{\partial\xi(x,z,\sigma+i\eta)}{\partial x}\in L^1(\mathbb{R}_\eta). \tag{50}$$

Function

$$\phi(x,z,t) = -\frac{1}{2\pi\sqrt{1-M^2}} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{d(\sigma+i\eta)x}$$

$$\times \left[\int_{-\infty}^{\infty} K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right) h(y,\sigma+i\eta)dy\right] e^{(\sigma+i\eta)t}d\eta$$

is then well defined for $z>0$, and doesn't depend on $\sigma \in [\sigma_1,\sigma_2]$.

Proof. To prove inclusion (49) of the Lemma it suffices to prove that under conditions above estimate

$$\left\| e^{d(\sigma+i\eta)x} \left[\int_{-\infty}^{\infty} K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right) h(y,\sigma+i\eta)dy\right]\right\|_{L^1_\eta(\mathbb{R})} < C(M,z) \tag{51}$$

holds uniformly with respect to $\sigma \in [\sigma_1,\sigma_2]$ for fixed x, fixed $z>0$, and for some $\sigma_1 > \sigma_a$. Applying then Theorem 47 from [Boc] we will obtain the second part of the Lemma.

Using asymptotics of $K_0(\zeta)$ for large and for small $|\zeta|$ ([EMOT]) we obtain the existence for fixed $z>0$ of a constant $A(z)>0$, large enough, such that estimates

$$\left|K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right)\right| < \frac{C(M,z)e^{-\sigma|x-y|}}{\sqrt{|\sigma+i\eta|\cdot|x-y|}} \text{ for } |x-y| > A(z), \tag{52}$$

and

$$\left|K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right)\right| < \frac{C(M,z)}{\sqrt{|\sigma+i\eta|}} \text{ for } |x-y| < A(z), \tag{53}$$

hold uniformly for $\sigma \in [\sigma_1,\sigma_2]$, with a constant C depending on M and z.

Combining estimates (52) and (53) with the estimate for $h(y,\lambda)$ we obtain

$$\left| e^{d(\sigma+i\eta)x} \left[\int_{-\infty}^{\infty} K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right) h(y,\sigma+i\eta)dy\right]\right|$$

$$< \frac{C(M,z)}{\sqrt{|\sigma+i\eta|}} \int_{-\infty}^{\infty} e^{-\sigma_1\cdot|y|}|h(y,\sigma+i\eta)|dy < \frac{C(M,z)}{(1+|\eta|)^{3+\epsilon}},$$

for $z>0$, which leads to estimate (51).

Again using estimates (52) and (53) and analogous estimates for

$$\frac{\partial}{\partial x}K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right),\ \frac{\partial^2}{\partial x^2}K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right),$$

and

$$\frac{\partial^2}{\partial z^2}K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right)$$

we obtain inclusions (50). □

To prove Theorem 1 we consider w_a satisfying condition (4), and define f_a by the formula (17). Using Lemma 7.1 we obtain that f_a satisfies estimate (45). Applying Proposition 5.3 to f_a

and using estimate (40) from Proposition 6.2 we obtain the existence of h_a satisfying equation (16) and such that

$$\|h_a(\cdot,\sigma+i\eta)\|_{\mathcal{L}^p(I^c(1))} < \exp\left\{e^{|\eta|}\cdot(1+|\eta|)^{2+\epsilon}\right\}\cdot\|f_a(\cdot,\sigma+i\eta)\|_{\mathcal{L}^2(I^c(1))} < \frac{C}{(1+|\eta|)^m}$$

for arbitrary m, arbitrary $p<\frac{4}{3}$, and $\sigma\in[\sigma_1,\sigma_2]$, with $\sigma_a<\sigma_1$.

Using the estimate above for $p=1$, we obtain

$$\int_{|x|>1}|h_a(x,\sigma+i\eta)|\cdot|x|^{-1}dx < \frac{C}{(1+|\eta|)^m}. \tag{54}$$

From the definition of h_a on $[-1,1]$ as

$$h_a(x,\lambda)=\frac{e^{-d(\lambda)x}\cdot\widehat{w_a}(x,\lambda)}{\pi}$$

and from condition (4) we obtain

$$\|h_a(x,\sigma+i\eta)\|_{L^p(I(1))} = \left\|e^{-d(\lambda)x}\cdot\widehat{w_a}(x,\sigma+i\eta)\right\|_{L^p(I(1))}$$

$$< C\,\|\widehat{w_a}(\cdot,\sigma+i\eta)\|_{L^2(I(1))} < \frac{C}{(1+|\eta|)^m}\ \text{for } p<\frac{4}{3},\ \sigma\in[\sigma_1,\sigma_2]\text{ with }\sigma_a<\sigma_1,$$

and therefore

$$\|h_a(\cdot,\sigma+i\eta)\|_{L^1(I(1))} < \frac{C}{(1+|\eta|)^m} \tag{55}$$

for arbitrary $m>0$.

From the estimates (54) and (55) we conclude that function h_a satisfies estimate (48), and therefore, applying Lemma 7.2 and Proposition 2.3, we obtain that function $\phi(x,z,t)$ in formula (5) is well defined and satisfies equation (1).

To prove that $\phi(x,z,t)$ satisfies boundary condition (2) we fix $x\in[-1,1]$ and denote $\delta=\min\{x+1,1-x\}$. Then we have

$$\lim_{z\to 0}\frac{\partial}{\partial z}\xi(x,z,\lambda)=\lim_{z\to 0}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}S(x-y,z,\lambda)e^{d(\lambda)y}h_a(y,\lambda)dy \tag{56}$$

$$=\lim_{z\to 0}\frac{\partial}{\partial z}\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}}S(x-y,z,\lambda)e^{d(\lambda)y}h_a(y,\lambda)dy$$

$$+\lim_{z\to 0}\frac{\partial}{\partial z}\int_{\mathbb{R}\setminus[x-\frac{\delta}{2},x+\frac{\delta}{2}]}S(x-y,z,\lambda)e^{d(\lambda)y}h_a(y,\lambda)dy.$$

For the first integral in the right hand side of (56) we obtain using Lemma 2.2

$$\lim_{z\to 0}\frac{\partial}{\partial z}\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}}S(x-y,z,\lambda)e^{d(\lambda)y}h_a(y,\lambda)dy$$

$$=-\lim_{z\to 0}\frac{\partial}{\partial z}\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}}e^{d(\lambda)y}h_a(y,\lambda)dy\int_{-\infty}^{\infty}e^{i(x-y)\omega}\frac{e^{-z\left((1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)\right)^{\frac{1}{2}}}}{2\sqrt{(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)}}d\omega$$

$$=\widehat{w_a}(x,\lambda).$$

For the second integral in the right hand side of (56) we have

$$\lim_{z\to 0}\frac{\partial}{\partial z}\int_{\mathbb{R}\setminus[x-\frac{\delta}{2},x+\frac{\delta}{2}]}S(x-y,z,\lambda)e^{d(\lambda)y}h_a(y,\lambda)dy$$

$$= -\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}} \lim_{z\to 0} \int_{\mathbb{R}\setminus[x-\frac{\delta}{2}, x+\frac{\delta}{2}]} \left[\frac{\partial}{\partial z} K_0\left(r(\lambda)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right)\right] h_a(y,\lambda)dy.$$

Using then estimate (54) and equality

$$\lim_{z\to 0}\left[\frac{\partial}{\partial z} K_0\left(r(\lambda)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{\frac{1}{2}}\right)\right] = 0$$

for $y \in \mathbb{R}\setminus[x-\frac{\delta}{2}, x+\frac{\delta}{2}]$ we obtain that the second integral in the right hand side of (56) is equal to zero.

From equalities above we conclude that

$$\lim_{z\to 0}\frac{\partial}{\partial z}\xi(x,z,\lambda) = \widehat{w_a}(x,\lambda)$$

for $x \in [-1,1]$ and $\mathrm{Re}\lambda \in [\sigma_1,\sigma_2]$, and therefore

$$\lim_{z\to 0}\frac{\partial}{\partial z}\phi(x,z,t) = w_a(x,t).$$

Straightforward substitution of $v_a(x,\lambda) = e^{d(\lambda)x}h_a(x,\lambda)$ into the formula (10), with $h_a(x,\lambda)$ defined as

$$h_a(x,\lambda) = \begin{cases} \dfrac{1}{\pi}e^{-d(\lambda)x}\cdot\widehat{w_a}(x,\lambda) \text{ for } x \in [-1,1], \\ \text{solution of equation (16) for } x \in \mathbb{R}\setminus[-1,1], \end{cases}$$

shows that $\xi(x,z,\lambda)$ defined by this formula satisfies equation (14) for $1 < |x| < A$. Then for $\phi(x,z,t)$ defined by formula (11) we will have

$$\begin{aligned} &\frac{\partial\phi(x,0,t)}{\partial t} + U\frac{\partial\phi(x,0,t)}{\partial x} \\ &= \frac{1}{2\pi}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\xi(x,0,\lambda)d\eta \\ &= \frac{1}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\left(\lambda + U\frac{\partial}{\partial x}\right)\xi(x,0,\lambda)d\eta = 0 \end{aligned}$$

for $1 < |x| < A$.

□

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